STABILITY STUDIES OF PULSE WIDTH MODULATED SYSTEMS

A Thesis presented to the
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ABSTRACT

In a general analysis of non-linear sampled data systems, one of the problems of paramount significance is that of stability, in particular, asymptotic stability in the large. For a pulse width modulated system, whose exact analysis is quite involved, the stability in the large of the system is established using

1) one of the theorems of stability in the Lyapunov's method,
2) describing function technique and
3) Popov's method.

In the Lyapunov's method, by assuming a positive definite quadratic function V as a Lyapunov's function, the sufficient condition is reduced to the negative definiteness of the difference ΔV of such a Lyapunov's function.

In the describing function technique, the system non-linearity is separated into multiple non-linearities of the saturation type connected in parallel and the sum of the describing function of each parallel loop gives the overall describing function of the pulse width modulated system.

To apply Popov's criterion for multi non-linear systems to pulse width modulated system, it is shown that the pulse width modulated system belongs to the class $\Gamma_n(\underline{K}, 0, \underline{K}')$ of the multi non-linear system. The theorem of the multi non-linear systems applicable to this class of systems is then applied to the pulse width modulated syutem. This method is found to be easier and the results obtained are comparable to the experimental values.

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M.S.Varadarajan.

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CHAPTER - I

INTRODUCTION

For the last two decades, sampled data systems have been extensively studied. Among the class of sampled data systems, the pulse width modulated system poses difficulties in terms of analytical investigation. In the pulse width modulated systems (hereafter referred to as PWM System), the output of the controller (Fig. 1) has fixed amplitude, but the duration of the pulse from the beginning of the sampling instant is proportional to the amplitude of the sampled signal. There are many types of PWM Systems. The lead and the lag type of PWM Systems are illustrated in Fig. 1. The PWM System was first used in the non-linear temperature regulating system by Gouy around 1897[1]. His regulating technique utilizes a sampling technique for controlling on a cyclic basis the time during which the relay contacts are closed and this time is inversely proportional to the temperature of the oven. PWM Systems have appeared in many forms and are used in many fields; for example, in hydraulic systems[2], minimal time control problems[3], and communication systems[4].

Study of PWM Systems, in general, is more difficult because of the presence of inherent non-linearities which arise out of the modulation scheme. The dynamic behaviour of a PWM System may be described by a set of non-linear difference equations and an exact solution is quite involved. Many approximate methods of analysis of PWM Systems exist. Amongst them, the techniques

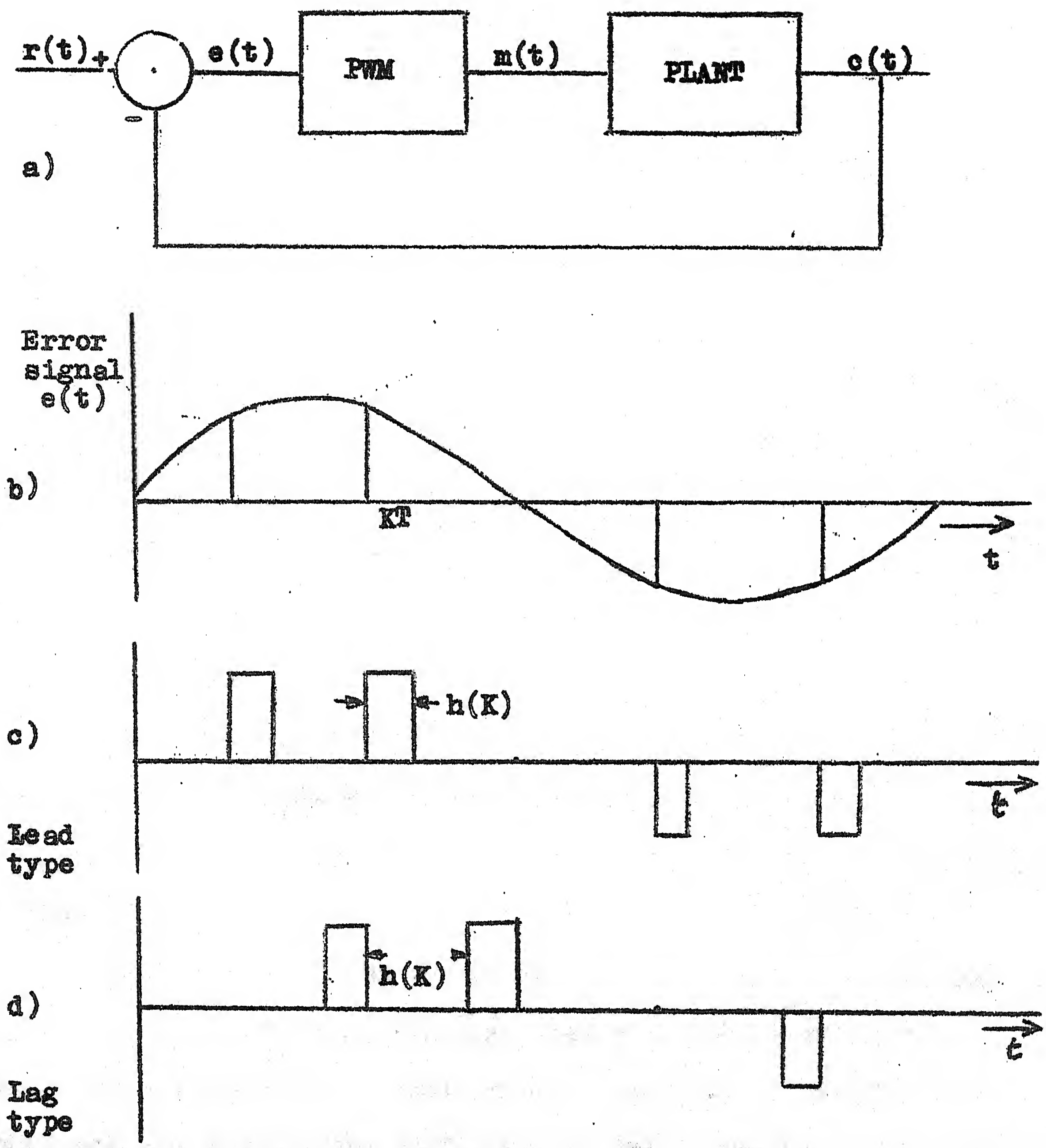


Fig.1 – Lead and Lag Type Pulse With Modulation.

based on Lyapunov's second method[5] and the describing function technique[6] are the most popular and useful. Recently Popov's method based on frequency domain concepts has been applied to non-linear sampled data systems[7-12].

Since in a general analysis of feedback control systems, one of the problems of paramount importance is that of stability, many simplified methods have been used to analyse the PWM System without actually solving the set of non-linear difference equations. The first paper in this field is due to Nease[13] who developed an approximate analysis and design procedure by linearising the non-linear system. Andeen[2] developed a simplified procedure for the analysis of a PWM System by replacing it by an equivalent pulse amplitude modulated system. The technique used herein was to approximate a non-linear system by a linear system.

An exact determination of the output response of a closed loop PWM System was developed by Delfeld and Murphy[6] who also extended the concept for the purpose of determining the existence of limit cycles of a PWM System based on describing function techniques. They had also shown that the PWM System could be represented as a multi non-linear system. Kadota and Bourne[5] have given a systematic procedure of obtaining the sufficient conditions for asymptotic stability in the large of a PWM System through the second method of Lyapunov. The Lyapunov's second method has been used mainly for system stability analysis as well as in choosing the system parameters. However, the results obtained through this method are conservative in nature compared

to actual experimental values giving rise to a large margin of safety. On the other hand, the describing function techniques used in examining the system for the existence of limit cycles, give pessimistic values for stability boundaries. Recently, Jury and Lee[14-15] have extended Popov's approach to the stability analysis of a multi non-linear sampled data systems.

The purpose of this investigation is to compare results obtained by the three methods viz., i) Lyapunov's second method, ii) Experimental Verification, and iii) Modified describing function techniques for systems of second and third order. Finally these results are compared to those obtained by the application of Popov's method. Investigations in references 5 and 6 were confined to second order systems. In the present investigation, systems of third order have been analyzed by the methods indicated above.

CHAPTER - II

LYAPUNOV'S STABILITY CRITERION APPLIED TO THE PWM SYSTEM

1. Preliminary Considerations:

This section is concerned with the stability of PWM Systems through the second method of Lyapunov which gives sufficient conditions for stability in terms of the given parameters of the system. The block diagram of a PWM System along with the type of output obtained for lead and lag type PWM is given in Fig. 1, where the linear plant is assumed to be time invariant. Only the lead type PWM System is considered here for the analytical treatment, which, however, could be extended to other types of PWM System.

Before obtaining the sufficient conditions for stability of a second and a third order systems, definitions of stability[16] and theorems in the second method of Lyapunov are reviewed for the sake of completeness.

2. Definitions of Stability and the Second Method of Lyapunov:

Let $t = KT$ be the K^{th} sampling instant and suppose that a discrete physical system is described by the variables $y_i = y_i(KT)$ satisfying the vector difference equation,

$$\underline{y}\left[(K+1)T\right] = \underline{Y}\left[\underline{y}(KT)\right] \qquad (2.2.1)$$

where $\underline{y}$ and $\underline{Y}$ are vectors with n components in the Euclidean space E_n,

$$\underline{y} = (y_1, y_2 .. y_n), \quad \underline{Y} = (Y_1, Y_2 .. Y_n).$$

Suppose that $\underline{y} = \underline{p}\,(KT)$ is a solution of (2.2.1) and let $\underline{x} = \underline{y} - \underline{p}(KT)$ be substituted in (2.2.1). Since $\underline{p}(KT)$ satisfies (2.2.1), the new vector $\underline{x}(KT)$ must satisfy the vector difference equation,

$$\underline{x}\,\left[(K+1)T\right] = \underline{X}\,\left[\underline{x}(KT)\right] \qquad (2.2.2)$$

where $\underline{X}$ is given by

$$\underline{X}\,\left[\underline{x}(KT)\right] = \underline{Y}\,\left[\underline{x}(KT) + \underline{p}(KT)\right] - \underline{Y}\,\left[\underline{p}(KT)\right] \qquad (2.2.3)$$

$\underline{x} = 0$, now represents a trivial solution of (2.2.2).

It is assumed that $\underline{X}\,\left[\underline{x}(KT)\right]$ is continuous in some region G of E_n for all $KT \geq 0$ and that a unique solution with the initial condition $\underline{x}(K_0T) = \underline{x}^0$ exists in the interval $KT \geq K_0T$.

<u>Definition 1</u>

The null solution $\underline{x} = 0$ of the system (2.2.2) is said to be <u>stable</u> provided that for an arbitrary $\varepsilon > 0$, there is a $d = d(\varepsilon, K_0T)$ such that whenever $\|\underline{x}^0\| < d$, the inequality

$$\left\|\underline{x}\;(K_0T,\ KT,\ \underline{x}^0)\right\| \leq \varepsilon \qquad \text{for all} \quad K \geq K_0.$$

<u>Definition 2</u>

The null solution $\underline{x} = 0$ of the system (2.2.2) is said to be <u>asymptotically stable</u> provided that the conditions of definition are satisfied and provided that

$$\lim_{K \to \infty} \underline{x}\,(K_0T,\ KT,\ \underline{x}^0) = 0.$$

<u>Definition 3</u>

The null solution $\underline{x} = 0$ of the system (2.2.2) is said to be <u>asymptotically stable in the large</u> provided that the conditions of definition 2 are satisfied and provided that d of definition 1

is allowed to be arbitrarily large.

The stability with which we are interested is asymptotic stability in the large as per definition 3 which for an autonomous system, physically means that for any large initial disturbance, the system comes back to the origin as $t \to \infty$. Lyapunov's theorem gives sufficient conditions for the stability of such systems.

<u>Theorem 1</u>[5]

If there exists in the whole space a function $V(\underline{x})$ (with the property that $V(\underline{0})=0$) which is definite and has the property $V(\underline{x}) \to \infty$ as $\underline{x} \to \infty$ and if ΔV is also a definite function having opposite sign to that of V, then the solution $\underline{x}=0$ of (2.2.2) is asymptotically stable in the large.

The difference of V is defined as

$$\Delta V = V\left[\underline{x}\,(K+1)T\right] - V\left[\underline{x}(KT)\right] \qquad (2.2.4)$$

3. <u>Stability Analysis of a Third Order PWM System</u>:

The treatment below follows that of Kadota and Bourne[5] except for the fact that we are considering a third order system. In the analysis that follows, a positive quadratic V function is chosen as Lyapunov's function and the sufficient conditions for asymptotic stability in the large of the trivial solution is obtained by putting ΔV as negative definite and constraining the given parameters of the PWM System.

The PWM modulator output $m(t)$ is defined by the equation,

$$\begin{aligned} m(t) &= A \operatorname{Sgn} e(KT), && KT \leq t < KT + b\,|e(KT)|\,, \ |e(KT)| \leq T/b \\ &= 0, && KT + b\,|e(KT)| \leq t < (K+1)T, \ |e(KT)| \leq T/b \\ &= A \operatorname{Sgn} e(KT), && KT \leq t < (K+1)T, \ |e(KT)| > T/b \end{aligned} \qquad (2.3.1)$$

where

$$\mathrm{Sgn}\,(x) = +1 \quad \text{for} \quad x > 0$$
$$\qquad\quad -1 \quad \text{for} \quad x < 0$$

and $e(KT)$ is the system error at $t = KT$, $K = 0, 1, \ldots$; T is the sampling period; b is the pulse width modulator constant, and A is the height of the output pulse.

The transfer function of the linear time invariant plant is represented by

$$G(s) = \frac{K}{(s-a_1)(s-a_2)(s-a_3)} \tag{2.3.2}$$

where K is a positive constant and a's are real and distinct. By introducing the following state variables $x_1 = e$, $x_2 = \dot{e}$, $x_3 = \ddot{e}$ and putting $u_3 = Km(t)$, equation (2.3.2) can be transformed into a set of first order differential equation of the form,

$$\dot{\underline{X}} = \underline{A}\,\underline{X} + \underline{U} \tag{2.3.3}$$

where

$$\underline{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_1a_2a_3 & -(a_1a_2+a_2a_3+a_3a_1) & (a_1+a_2+a_3) \end{bmatrix}$$

$$\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} ; \quad \underline{U} = \begin{bmatrix} 0 \\ 0 \\ u_3 \end{bmatrix}$$

Change of state variables is obtained by the transformation

$$\underline{X} = \underline{P}\,\underline{Y} \tag{2.3.4}$$

where P is given by[17]

$$\underline{P} = \begin{bmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ a_1^2 & a_2^2 & a_3^2 \end{bmatrix} \qquad (2.3.5)$$

Substituting (2.3.4) into (2.3.3), the following equations are obtained:

$$\dot{\underline{Y}} = \underline{J}\ \underline{Y} + \underline{V} \qquad (2.3.6)$$

where

$$\underline{J} = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$$

$$\underline{Y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} ; \qquad \underline{V} = \frac{Km}{D} \times \begin{bmatrix} a_3 - a_2 \\ a_1 - a_3 \\ a_2 - a_1 \end{bmatrix} \qquad (2.3.7)$$

and D is the determinant of $\underline{P}$ and is given by

$$D = a_3^2\,(a_2 - a_1) + a_2^2\,(a_1 - a_3) + a_1^2\,(a_3 - a_2).$$

The solution of (2.3.6) is given by

$$\underline{Y}(t) = e^{(t-t_o)\underline{J}}\, Y(t_0) + \int_{t_o}^{t} e^{(t-\tau)\underline{J}}\, \underline{V}(\tau)\,d\tau \qquad (2.3.8)$$

Substitute into (2.3.8) the following two sets of values of t_o and t :

$$t_o = KT,\ t = KT + h(K);\quad t_o = KT + h(K),\quad t = (K+1)T,$$

where $h(K)$ is the pulse width and is given by

$$h(K) = T \text{ Sat } \frac{b}{T} \left| e(KT) \right| ,$$

b is a positive constant and the saturation function is defined as

$$\text{Sat } x = \begin{cases} 1 & \text{for } x > 1 \\ x & \text{for } x \leq 1 \\ -1 & \text{for } x < -1. \end{cases}$$

The following equations are obtained:

$$\underline{Y}\left[KT+h(K)\right] = e^{h(K)\underline{J}}\,\underline{Y}(KT) + \int_{KT}^{KT+h(K)} e^{\left[KT+h(K)-\tau\right]\underline{J}}\,\underline{y}(\tau)\,d\tau$$

$$\underline{Y}\left[(K+1)T\right] = e^{\left[T-h(K)\right]\underline{J}}\,\underline{Y}\left[KT+h(K)\right]$$

Combining the above two equations,

$$\underline{Y}\left[(K+1)T\right] = e^{T\underline{J}}\,\underline{Y}(KT) + e^{\left[T-h(K)\right]\underline{J}}\int_{KT}^{KT+h(K)} e^{\left[KT+h(K)-\tau\right]\underline{J}}\,\underline{Y}(\tau)\,d\tau$$

Integrating and substituting (2.3.7), we obtain

$$y_i\left[(K+1)T\right] = e^{a_iT}\left[y_i(KT) + w_i(KT)\right] \qquad (2.3.9)$$

where

$$w_i(KT) = \frac{B_iA}{a_i}\left(e^{-a_iT\,\text{Sat}\,\frac{b}{T}\,y_s(KT)} - 1\right)\text{Sgn}\,y_s(KT) \quad ,(2.3.10)$$

$$\underline{B} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \frac{K}{D}\begin{bmatrix} a_3 - a_2 \\ a_1 - a_3 \\ a_2 - a_1 \end{bmatrix} = K\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix},$$

$$y_s(KT) = y_1(KT) + y_2(KT) + y_3(KT)$$

Equation (2.3.9) with (2.3.10) constitutes the description of the system.

Following the reference 5, we choose a Lyapunov's function of the form,

$$V = r_1 \, y_1^2 \, (KT) + r_2 \, y_2^2 \, (KT) + y_3^2 \, (KT) \qquad (2.3.11)$$

where r's are arbitrary positive constants.

Then

$$\Delta V = \sum_{\substack{i=1 \\ r_3=1}}^{3} r_i \left[e^{2a_i T} \, (y_i + w_i)^2 - y_i^2 \right] \qquad (2.3.12)$$

where $y_i(KT)$ and $w_i(KT)$ are abbreviated by y_i and w_i respectively.

Let $y_1 = R\,\psi_1$; $y_2 = R\,\psi_2$; $y_3 = R\,\psi_3$ (2.3.13)

where

$$\psi_1 = \text{Cos}\ \theta_1$$

$$\psi_2 = \text{Sin}\ \theta_1 \ \text{Cos}\ \theta_2$$

$$\psi_3 = \text{Sin}\ \theta_1 \ \text{Sin}\ \theta_2 \ \text{Cos}\ \theta_3 .$$

Then ΔV can be written as

$$\Delta V = R^2 \sum_{\substack{i=1 \\ r_3=1}}^{3} r_i \left[e^{2a_i T} \left(\psi_i + \frac{w_i}{R}\right)^2 - \psi_i^2 \right]$$

From the theorem stated in Section (2.2), the sufficient condition for asymptotic stability in the large of $\underline{Y} = 0$ of system (2.3.9) becomes

$$\sum_{\substack{i=1\\ r_3=1}}^{3} r_i \left[e^{2a_iT} z_i^2 - \Psi_i^2 \right] < 0 \qquad (2.3.14)$$

where Z_i is given by

$$Z_i = \Psi_i + \frac{B_i A \ \Psi}{a_i R \ |\Psi|} \left(e^{-a_iT \ \mathrm{Sat} \frac{b}{T} R |\Psi|} - 1 \right) \qquad (2.3.15)$$

The function Z_i is a piecewise monotonic function in R. This is monotonic between the following two intervals:

$$0 \leq R \leq \frac{T}{b \ |\Psi|}, \quad \frac{T}{b \ |\Psi|} \leq R < \infty \qquad (2.3.16)$$

Using the boundaries obtained in (2.3.16) equation (2.3.14) is reduced to the following:

$$\sum_{\substack{i=1\\ r_3=1}}^{3} r_i \ \theta_i \ (\Psi_i - B_i \ bA\Psi)^2 - \Psi_i^2 \ < 0$$

$$\sum_{\substack{i=1\\ r_3=1}}^{3} r_i \ \theta_i \ (\Psi_i + B_i bA \ \frac{e^{-a_iT} - 1}{a_iT} \Psi)^2 - \Psi_i^2 < 0$$

$$\sum_{\substack{i=1\\ r_3=1}}^{3} r_i \ (\theta_i - 1) \ \Psi_i^2 \ < 0 \qquad (2.3.17)$$

where

$$\theta_i = e^{2a_iT}; \quad B_i = KP_i ; \quad i = 1, 2, 3$$

Equations in (2.3.17) are in real quadratic forms and can be expressed as the following scalar products:

$$R^{-2}\,(\underline{Y}, \underline{BY}), \quad R^{-2}\,(\underline{Y}, \underline{CY})$$

where $\underline{B}$ and $\underline{C}$ are symmetric matrices whose elements are given by

$$B_{11} = d_o\, G^2 - 2\,\theta_1 r_1\, P_1\, G - (1-\theta_1)\, r_1$$

$$B_{12} = d_o\, G^2 - (r_1\theta_1 P_1 + r_2\theta_2 P_2)\, G$$

$$B_{13} = d_o\, G^2 - (r_1\theta_1 P_1 + \theta_3 P_3)\, G$$

$$B_{22} = d_o\, G^2 - 2\,\theta_2 r_2 P_2 G - (1-\theta_2)\, r_2$$

$$B_{23} = d_o\, G^2 - (r_2\theta_2 P_2 + \theta_3 P_3)\, G$$

$$B_{33} = d_o\, G^2 - 2\,\theta_3 P_3 G - (1-\theta_3)$$

$$C_{11} = d_1 G^2 + 2\,\theta_1\, r_1\, P_1\, Q_1\, G - (1-\theta_1)\, r_1$$

$$C_{12} = d_1 G^2 + (r_1\, \theta_1\, P_1\, Q_1 + r_2\, \theta_2 P_2\, Q_2)\, G$$

$$C_{13} = d_1 G^2 + (r_1\, \theta_1\, P_1\, Q_1 + \theta_3\, P_3\, Q_3)\, G$$

$$C_{22} = d_1 G^2 + 2\theta_2\, r_2\, P_2\, Q_2\, G - (1-\theta_2)\, r_2$$

$$C_{23} = d_1 G^2 + (r_2\, \theta_2\, P_2\, Q_2 + \theta_3\, P_3\, Q_3)\, G$$

$$C_{33} = d_1 G^2 + 2\theta_3\, P_3\, Q_3\, G - (1-\theta_3)$$

where

$$\theta_i = e^{2a_i T}\;; \quad Q_i = \frac{e^{-a_i T} - 1}{a_i T}\;; \quad i = 1, 2, 3$$

$$G^{*} = KAb$$

---

* G is defined as the loop gain of the non-linear system.

$$d_1 = r_1 \theta_1 P_1^2 Q_1^2 + r_2 \theta_2 P_2^2 Q_2^2 + \theta_3 P_3^2 Q_3^2$$

$$d_o = r_1 \theta_1 P_1^2 + r_2 \theta_2 P_2^2 + \theta_3 P_3^2$$

Then, from the property of real quadratic forms, the first two conditions of (2.3.17) are reduced to all the eigen values of matrices $\underline{B}$ and $\underline{C}$ being negative. The third condition of (2.3.17) refers simply that a_1, a_2 and a_3 are all less than zero.

Hence for asymptotic stability in the large of $\underline{Y} = 0$ of systems (2.3.9), all the eigen values of matrices $\underline{A}$, $\underline{B}$ and $\underline{C}$ are negative.

To determine the constants r_1 and r_2, we use the critical loop gain*. From the conditions on the eigen values of the matrix $\underline{C}$, we obtain[18]

$$\begin{aligned} &C_{11} < 0 \\ &C_{22} < 0 \\ &C_{33} < 0 \\ &C_{12}^2 - C_{11} C_{22} < 0 \end{aligned}$$

and

$$\text{Det } (C) < 0 \qquad (2.3.18)$$

Solving for the first four inequalities of (2.3.18), we have

$$e_1 < \frac{-\theta_1 r_1 P_1 Q_1}{d_1} - \left[\left(\frac{\theta_1 r_1 P_1 Q_1}{d_1} \right)^2 + \frac{(1-\theta_1) r_1}{d_1} \right]^{\frac{1}{2}}$$

* Critical loop gain is defined as the loop gain KAb of the system for which it goes unstable.

$$G_2 < \frac{-\theta_2 r_2 P_2 Q_2}{d_1} - \left[\left(\frac{\theta_2 r_2 P_2 Q_2}{d_1}\right)^2 + \frac{(1-\theta_2) r_2}{d_1} \right]^{\frac{1}{2}}$$

$$G_3 < \frac{-\theta_3 P_3 Q_3}{d_1} - \left[\left(\frac{\theta_3 P_3 Q_3}{d_1}\right)^2 + \frac{(1-\theta_3)}{d_1} \right]^{\frac{1}{2}}$$

$$G_4 < \frac{-d_2}{d_3} - \left[\left(\frac{d_2}{d_3}\right)^2 + \frac{(1-\theta_1)\,(1-\theta_2) r_1 r_2}{d_3} \right]^{\frac{1}{2}} \qquad (2.3.19)$$

where

$$d_2 = \theta_1\, r_1\, P_1\, Q_1\, r_2\, (1-\theta_2) + \theta_1\, r_2\, P_2\, Q_2\, r_1\, (1-\theta_1)\ ,$$

$$d_3 = (r_1\, \theta_1\, P_1\, Q_1 - r_2\, \theta_2\, P_2\, Q_2\,)^2 + d_1 \left[(1-\theta_1) r_1 + (1-\theta_2) r_2 \right]$$

From the last inequality of (2.3.18), G_5 is obtained. Furthermore, $\underline{B}$ matrix is obtained from $\underline{G}$ matrix by setting $Q_1 = Q_2 = Q_3 = 1$. Hence these inequalities are also implied by (2.3.19). For all positive G's, the critical loop gain $(KAb)_c$ becomes the minimum of $G_i(r_1, r_2)$ which is maximum with respect to $r_j (j = 1,2)$; that is

$$(KAb)_c = \frac{\max}{r_j} \quad \frac{\min}{i}\, G_i\,(r_j) \qquad i = 1,2, \ldots 5,$$

$$j = 1,2$$

$$\text{for all} \quad r > 0 \qquad (2.3.20)$$

4.a <u>Examples</u>:

The examples considered are

$$G(s) = \frac{K}{s(s+1)} \qquad (2.4.1)$$

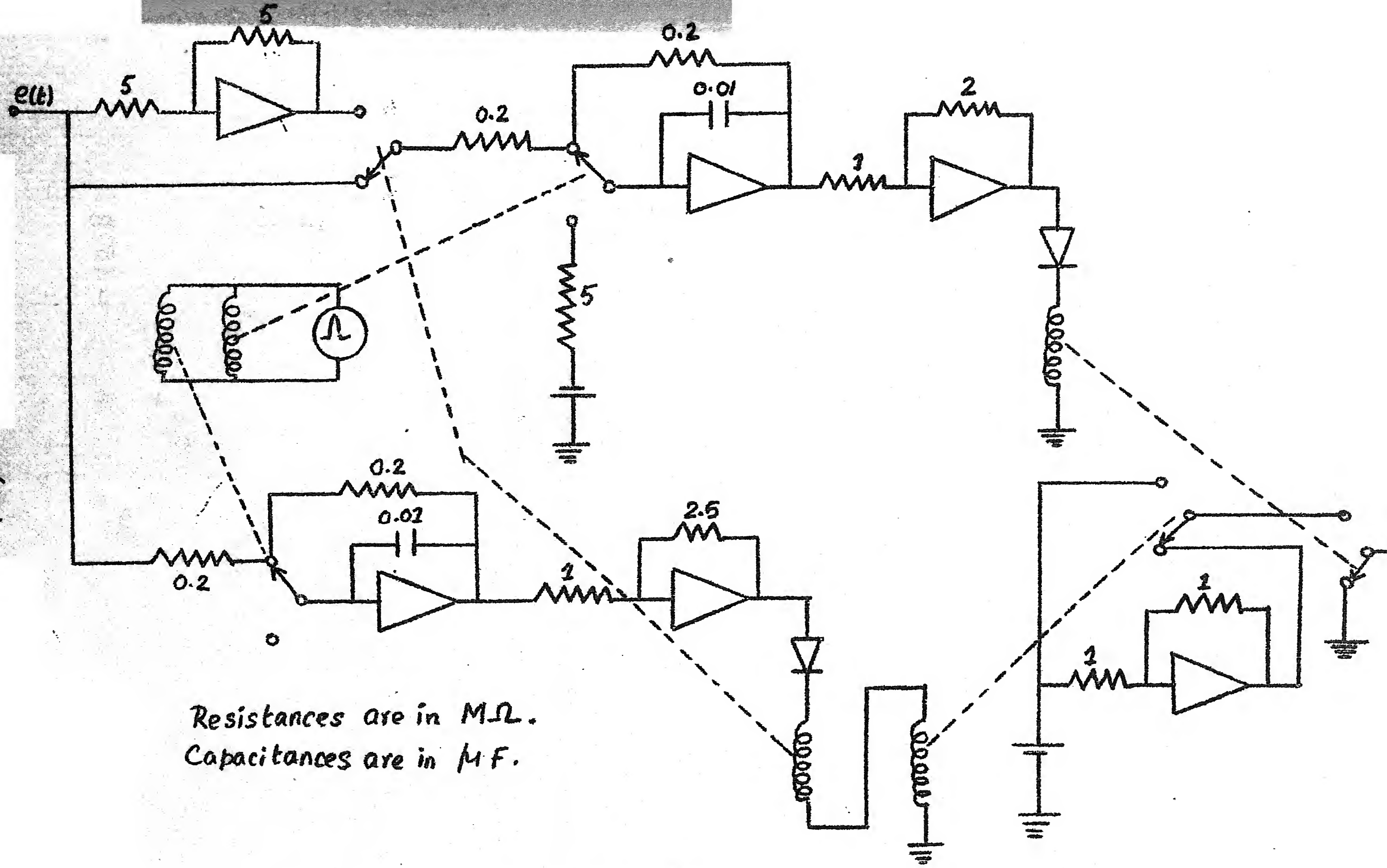
for the second order time invariant plant and

$$G(s) = \frac{K}{s(s+1)(s+2)} \qquad (2.4.2)$$

for the third order system. For the calculation of $(KAb)_c$ in (2.3.20) the computations involved are more in the case of third order system because G_i $(i = 1, 2, ..5)$ and r_j $(j = 1,2)$ are involved. For the second order case hand computation may be sufficient. However, a digital computer programme was written for both second and third order cases for the calculations of $(KAb)_c$ and the results are given by Curve (1) presented in Figs. 6 and 7 for different values of sampling frequencies.

4.b <u>Experimental Set up</u>:

The above systems were simulated on an analog computer. The connection diagram of PWM is given in Fig. 2. For each sampling frequency, the gain was varied till the system became unstable. The results are plotted in Curve (3) in Figs. 6 and 7.



Resistances are in MΩ.
Capacitances are in μF.

Fig. 2. – Pulse Width Modulator

CHAPTER - III

STABILITY ANALYSIS OF PWM SYSTEMS USING DESCRIBING FUNCTION METHOD

1. Representation of PWM System as a Multi Non-Linear System:

A little reflection on equation (2.3.9) of the preceding Section tells us that $y_s(KT) = \sum_1^n y_i(KT)$ can be considered as a sum of n outputs for a general system with n non-linearities, each from a non-linear system. This would enable us to represent the PWM System as a multi non-linear system.

From equation (2.3.9)

$$y_i\,(K+1)T = e^{a_iT}\,y_i(KT) + \frac{B_iA}{a_i}\,e^{a_iT}\left(e^{-a_ih(K)}-1\right)\,\mathrm{Sgn}\,e(KT),\quad i = 1,2\cdots n.$$

where $e(KT) = \sum_1^n y_i\,(KT)$.

Writing this as a vector difference equation, we obtain

$$\underline{Y}\,[(K+1)T] = \underline{G}(T)\,\underline{Y}(KT) + \underline{B}'\,A\,b\,\underline{V}\,(KT) \qquad (3.1.1)$$

where vector $\underline{V}$ is defined as

$$\underline{V} = \begin{bmatrix} V_1 \\ V_2 \\ \cdot \\ \cdot \\ \cdot \\ V_n \end{bmatrix} \qquad (3.1.2)$$

$$\underline{G}(T) = \begin{bmatrix} e^{a_1T} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & e^{a_2T} & & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & e^{a_nT} \end{bmatrix} \qquad (3.1.3)$$

$\underline{B}'$ is the transpose of $\underline{B}$ and

$$V_i(KT) = e^{a_i T} \left[\frac{e^{-a_i h(K)} - 1}{a_i b} \right] \text{Sgn } e(KT). \qquad (3.1.4)$$

Taking the Z-transform of (3.1.1), we get

$$Z\underline{Y}(Z) = \underline{G}(T)\, \underline{Y}(Z) + \underline{B}'\, A\, b\, \underline{V}(Z) \qquad (3.1.5)$$

Provided that $\underline{Y}(0) = 0$. The output of the system is given by

$$\underline{Y}(Z) = \left[Z\underline{I} - \underline{G}(T)\right]^{-1} \underline{B}'\, A\, b\, \underline{V}(Z). \qquad (3.1.6)$$

Simplifying (3.1.6), we obtain

$$Y(Z) = Ab \sum_{i=1}^{n} \frac{B_i\, V_i(Z)}{Z - e^{a_i T}}. \qquad (3.1.7)$$

The block diagram based on (3.1.7) is shown in Fig. 3.

2. __Stability of PWM System:__

It can be shown that the describing function for each non-linear element[6] is given by

$$\begin{aligned} N_i(E_m) &= \frac{V_{im}}{E_m} \\ &= e^{a_i T} \left[1 - .425(a_i b E_m) + .125(a_i b E_m)^2 - \right. \\ &\quad \left. .0265(a_i b E_m)^3 + .0104(a_i b E_m)^4 - \ldots \right] \\ &\qquad \text{for } E_m \leq T/b, \end{aligned} \qquad (3.2.1)$$

where

$$e(t) = E_m \text{ Sin } wt, \qquad (3.2.2)$$

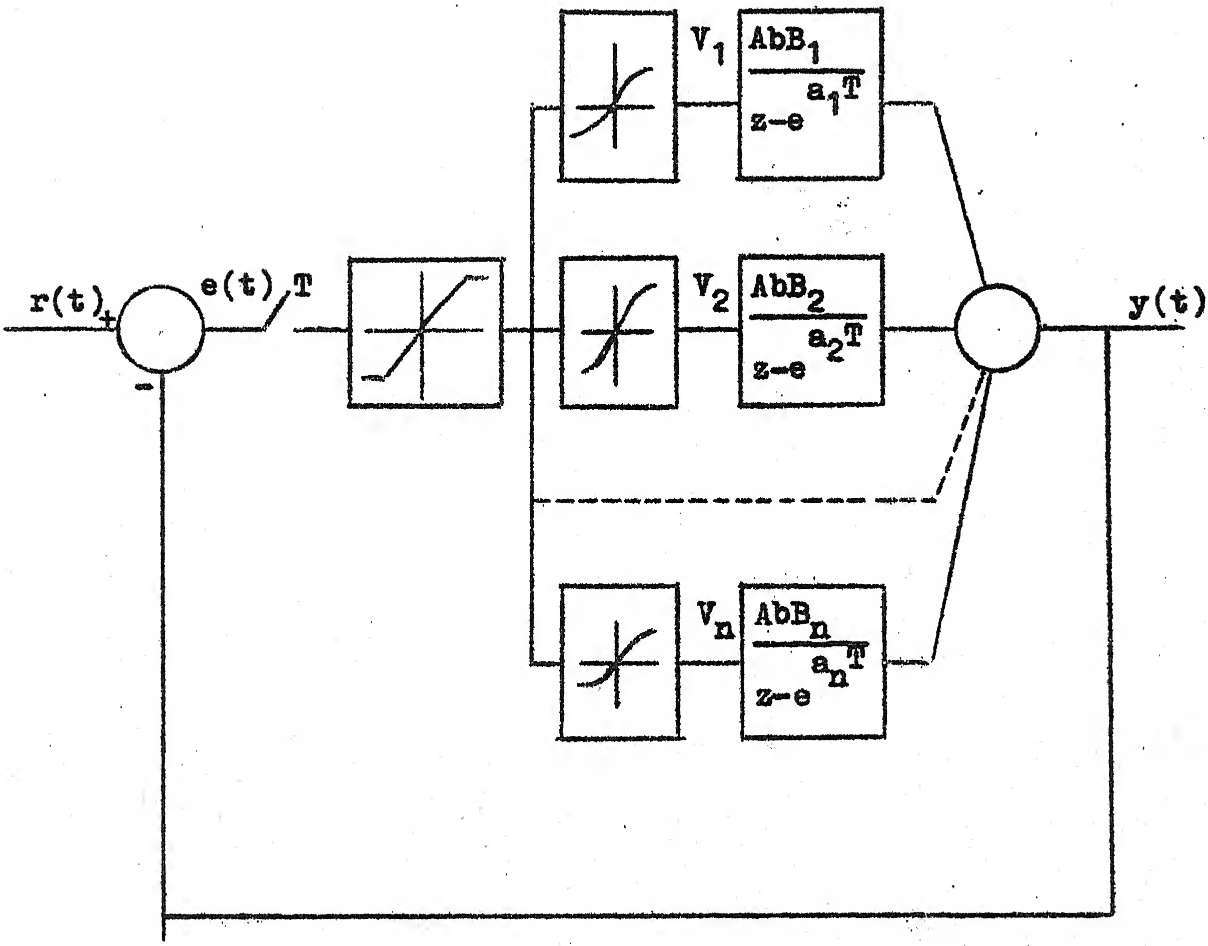


FIG. 3: MULTI NON-LINEAR REPRESENTATION OF PWM.

$$V_1(t) = V_{1m} \text{ Sin } wt + \sum_{k=2}^{\infty} C_k \text{ Sin } k\, wt \qquad (3.2.3)$$

The derivation of the above and the following is found in(6).

For magnitudes of E_m greater than the saturation level, the describing function is approximated by

$$N_1(E_m) = \frac{2}{\pi} \left[(wt_o + \frac{\text{Sin } 2\, wt_o}{2}) - (1-e^{a_1 T}) \times (wt_1 + \frac{\text{Sin } 2\, wt_1}{2}) \right], \quad E_m > T/b, \qquad (3.2.4)$$

where

$$wt_o = \text{Sin}^{-1} \left(\frac{T/b}{E_m}\right)$$

and

$$wt_1 = \text{Sin}^{-1} \left[\frac{1 + (\frac{1-e^{a_1 T}}{a_1 T})}{E_m} \right], \qquad (3.2.5)$$

$$\text{For } E_m > \left[1 + \frac{(1-e^{a_1 T})}{a_1 T} \right] T/b. \qquad (3.2.6)$$

From Fig. 3, for an autonomous system

$$Y(jw) = Ab \sum_{i=1}^{n} \frac{B_1 N_1 (E_m)}{e^{jwT} - e^{a_1 T}} \times E(jw) \qquad (3.2.7)$$

and

$$Y(jw) = -E(jw) \qquad (3.2.8)$$

From equations (3.2.7) and (3.2.8),

$$1 + G\,(jw, E_m) = 0 \qquad (3.2.9)$$

where

$$G(jw,\ E_m) = Ab \sum_{i=1}^{n} \frac{B_i\ N_i\ (E_m)}{e^{jwT} - e^{s_i T}} \ . \tag{3.2.10}$$

Stability boundaries may be obtained by plotting a family of Curves of $G(jw,\ E_m)$ near the critical point (-1,0).

3. <u>Digital Computer Solution of Second and Third Order Systems:</u>

The systems considered are

$$G(s) = \frac{K}{s(s+1)} \tag{3.3.1}$$

for which $$G(jw,\ E_m) = KAb \left[\frac{N_1\ (E_m)}{e^{jwT} - 1} - \frac{N_2\ (E_m)}{e^{jwT} - e^{-T}} \right] \tag{3.3.2}$$

and

$$G(s) = \frac{K}{s(s+1)\ (s+2)} \tag{3.3.3}$$

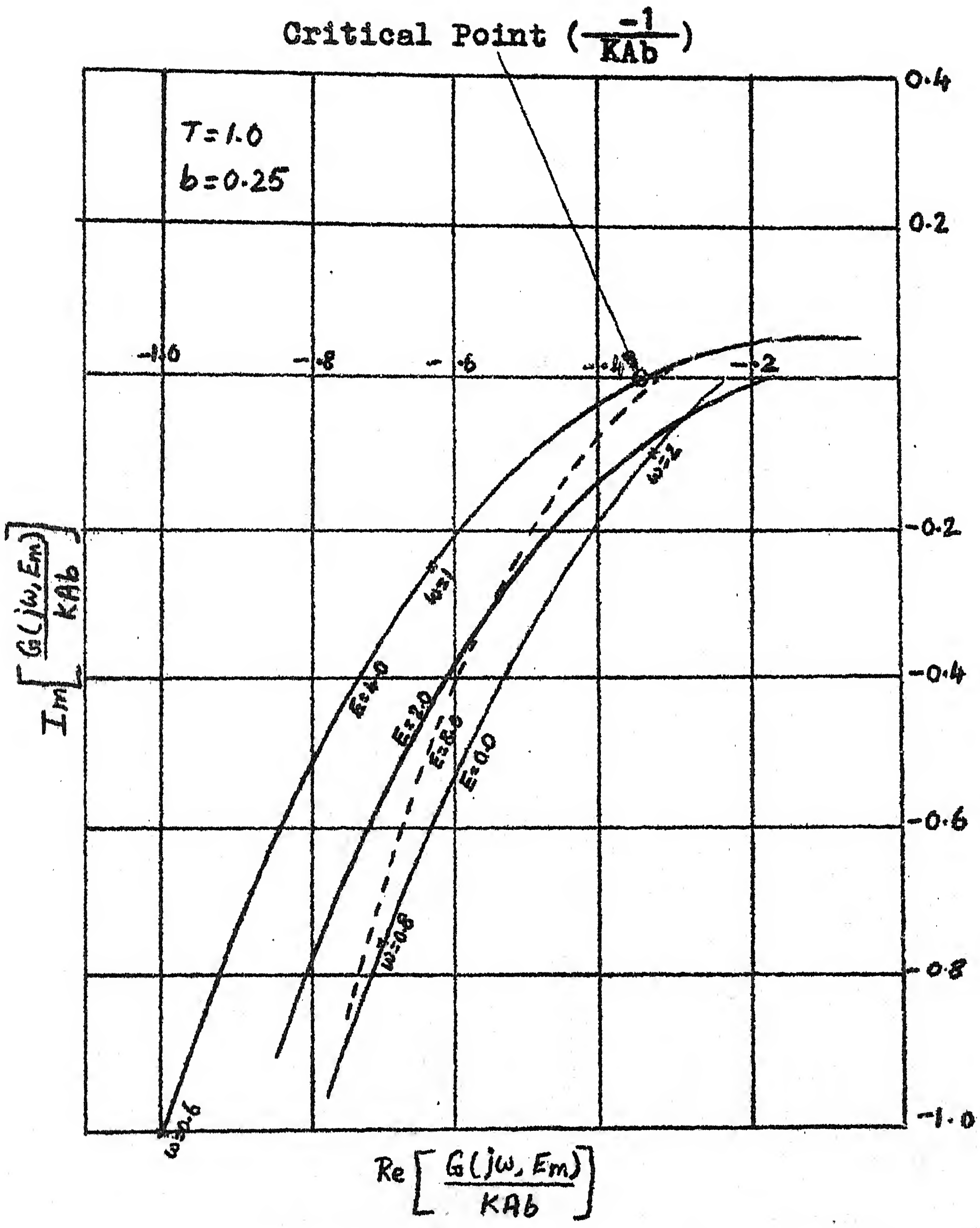
with

$$G(jw,\ E_m) = KAb \left[\frac{N_1\ (E_m)}{2(e^{jwT} - 1)} - \frac{N_2\ (E_m)}{e^{jwT} - e^{-T}} + \frac{N_3\ (E_m)}{2(e^{jwT} - e^{-2T})} \right] . \tag{3.3.4}$$

In equations (3.3.2) and (3.3.4), $N_i(E_m)$, (i=1,2,3) are obtained from equations (3.2.1) through (3.2.6).

Equations (3.3.2) and (3.3.4) were solved on IBM 7044. The general purpose programme for the describing function method of analysis of a PWM System is given in Appendix 1. The typical Curves plotted for a particular sampling frequency are shown in Fig. 4 for the second order system and Fig. 5 for the third order system. The critical loop gain and the frequency of oscillation

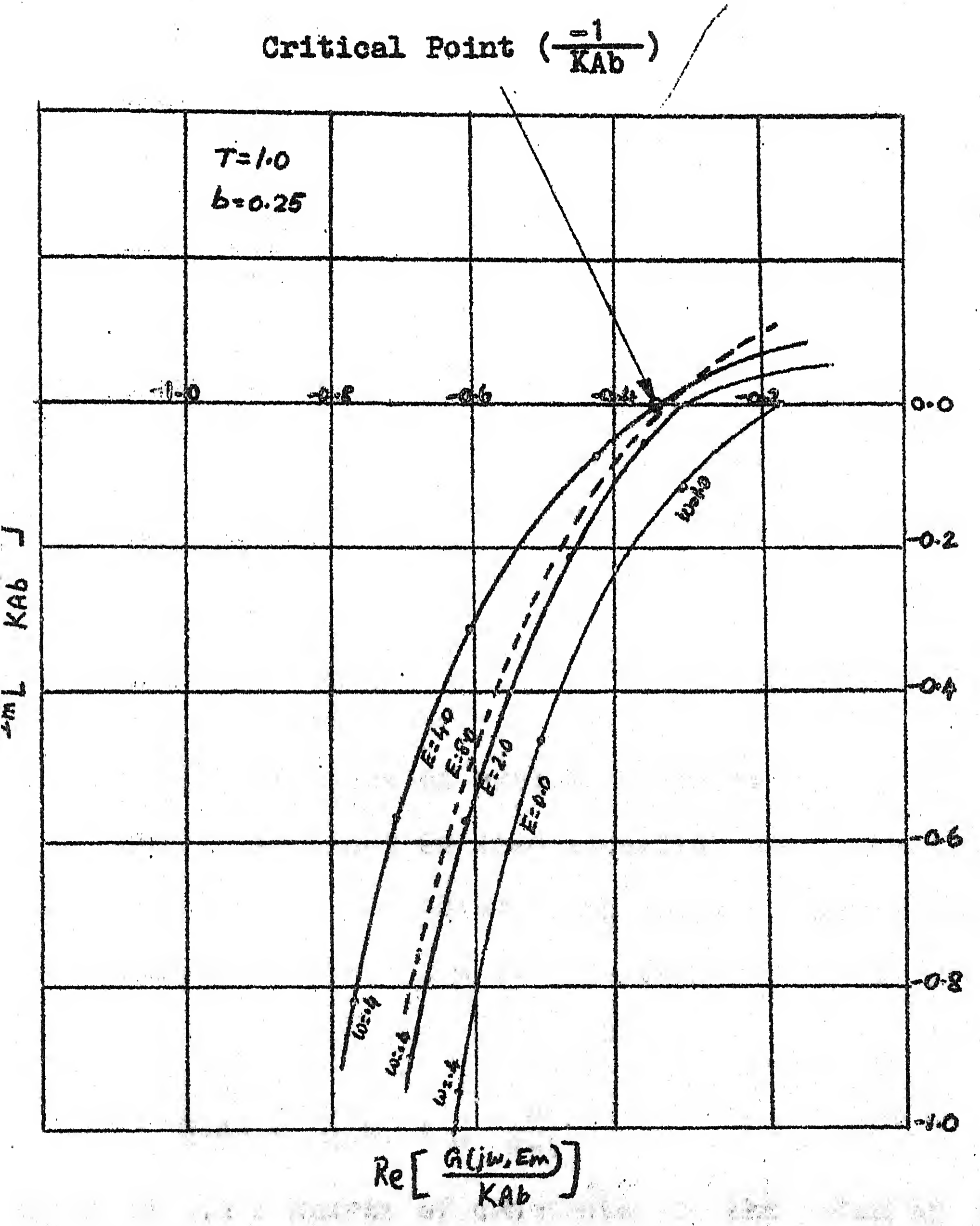
are obtained from the intersection of the Curve for $E_m = T/b$ with the negative real axis. These Curves are repeated varying the sampling period T and the results are plotted as a function of frequency as shown in Figs. 6 and 7 respectively for the second and third order systems.



——— below saturation

- - - - above saturation

FIG. 4: NYQUIST PLOT FOR A SECOND ORDER PWM SYSTEM



—— below saturation,
– – – above saturation.

FIG. 5: NYQUIST PLOT FOR A THIRD ORDER PWM SYSTEM.

CHAPTER - IV

ABSOLUTE STABILITY OF PWM SYSTEM THROUGH POPOV'S CRITERION

1. Definition and Description of the System:

Popov's method to the absolute stability of non-linear discrete systems has been extended to multiple non-linear discrete systems by Jury and Lee[14-15] and the approach is essentially in frequency domain. This extension of the stability criterion to PWM is considered in this Section and the stability boundaries obtained for second and third order systems are compared with the boundaries obtained in the methods of the preceding Sections. It can be seen that the formulation presents no difficulty since the PWM has been represented in Section (3.1) as a parallel system with many non-linearities.

For the type of system considered here, it is assumed that the linear plant is time invariant and has a transfer function $G_{ij}(z)$ with all its poles lying inside a unit circle except for a simple pole at $z = 1$. The transfer matrix can be written as

$$\underline{G}^*(z) = \underline{G}_1^*(z) + \underline{R}\,\frac{z}{z-1} \tag{4.1.1}$$

where $\underline{R}$ is a matrix of constants and the poles of $\underline{G}_1^*(z)$ lie inside the unit circle. For an autonomous system, the vector $\underline{e}(KT)$ which is an input to the non-linear elements is related to the system variable by the equation[14]

$$\underline{e}(KT) = \underline{f}(0,KT) - \sum_{l=0}^{KT} g(KT-l)\ \underline{v}(l) - \underline{R}\ \underline{P}(KT) \qquad (4.1.2)$$

where $\underline{f}(0,KT)$ is the response due to the initial conditions, $\underline{v}(l)$ is a n vector whose components are the outputs of n-non-linear elements and $\underline{P}(KT)$ is the response due to the pole at $z=1$. $\underline{g}(KT)$ is the inpulse response matrix corresponding to the poles inside the unit circle.

Definition 1

Any discrete system with non-linear elements $\underline{V}(KT)$ coupled to a stable linear plant described by the transfer matrix (4.1.1) and having a response given by (4.1.2) is said to belong to the class $\Gamma_n(\underline{K})$, $\underline{K}$ (K_{ii}), $K_{ii} > 0$ $(i=1,2\ ..n)$ such that

i) $v_i\ (0) = 0$, $(i=1,2\ ..n)$,

ii) $0 \leq \dfrac{v_i\ (e_i)}{e_i} < K_{ii}$,

iii) $\underline{v}(KT) \rightarrow 0$ as $\underline{e}(KT) \rightarrow 0$.

Definition 2

Any discrete system satisfying the conditions given in definition 1 is said to belong to the class $\Gamma_n(\underline{K}, \underline{K}', \underline{K}'')$ with

$$\underline{K} = (K_{ii})$$

$$\underline{K}' = (K'_{ii})$$

$$\text{and} \quad \underline{K}'' = (K''_{ii})$$

provided that the non-linearities satisfy the condition on a.c. gain; namely

$$K_{ii}' < \frac{dv_i(e_i)}{de_i} < K_{ii}''$$

for $i = 1, 2 \ldots n$.

Among the various systems belonging to the class $\Gamma_n(\underline{K}, \underline{K}', \underline{K}'')$, we are mainly interested in the system shown in Fig. 3.

2. Absolute Stability of PWM System:

The absolute stability of a PWM System is based on theorems on stability of a multi non-linear system[14]. The theorems are only stated here for the sake of completeness.

Theorem 1

The null solution of any system belonging to the class $\Gamma_n(\underline{K}, -\underline{K}', \underline{K}')$ is absolutely stable if there exists a $\underline{Q}$ matrix with real diagonal elements such that

$$\underline{H}^*(z) = 2\underline{K}^{-1} + \underline{Y}_1^*(z) + \overline{\underline{Y}_1^*(z)}^T > 0, \quad |z| = 1, \qquad (4.2.1)$$

where

$$\underline{Y}_1^*(z) = \left[\underline{I}+(z-1)\underline{Q}(z)\right]\underline{G}^*(z) - \tfrac{1}{2}\,|z-1|^2\,\overline{\underline{G}^*(z)}^T\,|\underline{Q}|\,\underline{K}'\underline{G}^*(z), \qquad (4.2.2)$$

and

$$\underline{Q}(z) = q_{ii}(z) \quad \text{with}$$

$$q_{ii}(z) = \begin{cases} q_{ii} & \text{if } q_{ii} < 0, \\ q_{ii}\, z^{-1} & \text{if } q_{ii} \geq 0. \end{cases}$$

Theorem 2

The null solution of any system belonging to $\Gamma_n(\underline{K}, 0, \infty)$ is absolutely stable if

$$\underline{H}^*(z) = 2\underline{K}^{-1} + \underline{Y}_2^*(z) + \overline{\underline{Y}_2^*(z)}^T > 0, \quad |z| = 1, \tag{4.2.3}$$

where

$$\underline{Y}_2^*(z) = \boxed{\underline{I} + (z-1)\underline{Q}(z)} \quad \underline{G}^*(z) \tag{4.2.4}$$

and $Q(z)$ is as defined in theroem 1.

The two theorems given above are quite general for their applicability to any multi non-linear sampled data system. It will be shown that the PWM system belongs to the class $\Gamma_n(\underline{K}, 0, \underline{K}')$ and the following theorem[15], applicable only to saturation types of multi non-linearities proves to be more useful to obtain stability boundaries very close to the one obtained experimentally.

Theorem 3

The null solution of any system belonging to $\Gamma_n(\underline{K}, 0, \underline{K}')$ is absolutely stable if

$$\underline{H}^*(z) = 2\underline{K}^{-1} + \underline{Y}_3^*(z) + \overline{\underline{Y}_3^*(z)}^T > 0, \quad |z| = 1, \tag{4.2.5}$$

where

$$\underline{Y}_3^*(z) = \boxed{\underline{I} + (z-1)\underline{Q}(z)} \quad \underline{G}^*(z) + \tfrac{1}{2}|z-1|^2 \quad \underline{Q} \quad \underline{K}'^{-1} \tag{4.2.6}$$

and $\underline{Q}(z)$ is as defined earlier.

In all the theorems given above $\overline{\underline{Y}_i^*(z)}^T$, $(i=1,2,3)$ represents the complex conjugate transpose of $\underline{Y}_i^*(z)$.

For an equivalent PWM System shown in Fig.3,

$$
\begin{aligned}
m(t) &= A, \quad KT \leq t < KT + b\,|e(KT)| \qquad |e(KT)| \leq T/b \\
&= 0, \quad KT + b\,|e(KT)| \leq t < (K+1)T \\
&= A, \quad KT \leq t < (K+1)T, \quad |e(KT)| > T/b
\end{aligned}
\tag{4.2.7}
$$

The transfer function is given by

$$G(s) = \frac{R_1}{s} + \sum_{i=2}^{n} \frac{R_i}{(s-a_i)}$$

from which the transfer matrix $\underline{G}^*(z)$ can be written as

$$\underline{G}^*(z) = G^*_{ij}(z) = \frac{R_i}{z-e^{a_j T}} \qquad (4.2.8)$$

$i,j = 1, 2, \ldots n$ and $a_j = 0$ for $j = 1$.

For the system given in Fig.3,

$$e_i(KT) = e(KT)$$

$$\begin{aligned} |V_i(e_i)| = |V_i(e)| &= \frac{-A}{a_i}\, e^{a_i T} \; e^{-a_i be(KT)} - 1, \quad |e(KT)| \leq T/b \\ &= \frac{-A}{a_i}\,(1-e^{a_i T}), \qquad\qquad |e(KT)| > T/b \end{aligned} \qquad (4.2.9)$$

$$K'_{ii} = \left|\frac{dV_i(e_i)}{de_i}\right| = \left|\frac{dV_i(e)}{de}\right| = Ab \qquad (4.2.10)$$

Inspection of equations (4.2.9) and (4.2.10) shows that PWM System belongs to the class $\sqrt{n}(\underline{K}, 0, \underline{K}')$ and the theorem 3 stated earlier can be applied straight to find the stability boundaries of the system. This is illustrated for a second and a third order systems in the next section.

3. Examples

a) Second Order System:

The transfer function of the linear time invariant plant is given by

$$G(s) = \frac{K}{s(s+1)} \qquad (4.3.1)$$

Taking z transforms,

$$G^*(z) = \frac{K}{z-1} - \frac{K}{z-p} \tag{4.3.2}$$

where $p = e^{-T}$.

The transfer matrix is given by

$$\underline{G}^*(z) = \begin{bmatrix} \frac{K}{z-1} & \frac{-K}{z-p} \\ \frac{K}{z-1} & \frac{-K}{z-p} \end{bmatrix} \tag{4.3.3}$$

Using (4.2.5) and (4.2.6) it is shown in Appendix 2 that

$$\underline{H}^*(z) = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} > 0, \text{ for all } |z| = 1, \tag{4.3.4}$$

where the elements of the matrix $\underline{H}^*(z)$ are given by

$$H_{11} = \frac{2}{KAb} + \frac{2Q_1}{KAb}(1-\cos\theta) - 1 + 2Q_1 \cos\theta,$$

$$H_{22} = \frac{2T}{KAb(1-p)} + \frac{2Q_2(1-\cos\theta)}{KAb} - Y\left[2Q_2\sin^2\theta + 2(\cos\theta - p)\right.$$
$$\left.\{1+Q_2(1-\cos\theta)\}\right]$$

where $Y = \dfrac{1}{1+p^2-2p\cos\theta}$

$$H_{12} = (\cos\theta + j\sin\theta)Q_2 + \frac{\cos\theta - 1 + j\sin\theta}{2(1-\cos\theta)} -$$
$$(\cos\theta - p - j\sin\theta)Y\left[1+Q_1(1-\cos\theta + j\sin\theta)\right]$$

and $H_{21} = \overline{H_{12}}$.

It is established[14] that $\underline{H}^*(z)$ is a positive definite matrix and therefore, for the inequality (4.3.4) to be satisfied, the necessary conditions[18] are that

$$H_{11} > 0,$$
$$H_{22} > 0,$$
$$\begin{vmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{vmatrix} > 0. \qquad (4.3.5)$$

Expanding the determinant (4.3.5), we obtain a quadratic expression in (KAb) which is the loop gain of the system and this should be maximized with respect to Q_1 and Q_2. Thus the critical loop gain is given by

$$(KAb)_c = \max_{Q_j} (KAb) \text{ such that } |H^*(z)| > 0, \; j = 1, 2.$$

It is to be noted that in the absence of any constraint on the a.c.gain of the non-linearities, the system reduces to the class $\Gamma_n(\underline{K}, 0, \infty)$ and the enequality (4.3.4) becomes

$$\begin{bmatrix} H_{11} - \dfrac{2Q_1(1-\cos\theta)}{KAb} & H_{12} \\ H_{22} & H_{22} - \dfrac{2Q_2(1-\cos\theta)}{KAb} \end{bmatrix} > 0, \qquad (4.3.6)$$

for all $|z| = 1$.

The critical loop gains obtained from (4.3.4) and (4.3.6) for different frequencies are plotted in Fig.6.

b) <u>Third Order System:</u>

The system transfer function is given by

$$G(s) = \frac{K}{s(s+1)(s+2)} \qquad (4.3.7)$$

Taking the z transform, the transfer matrix $\underline{G}^*(z)$ can be written as

$$\underline{G}^*(z) = \begin{bmatrix} \frac{0.5K}{z-1} & \frac{-K}{z-p} & \frac{0.5K}{z-q} \\ \frac{0.5K}{z-1} & \frac{-K}{z-p} & \frac{0.5K}{z-q} \\ \frac{0.5K}{z-1} & \frac{-K}{z-p} & \frac{0.5K}{z-q} \end{bmatrix} \qquad (4.3.8)$$

where $p = e^{-T}$ and $q = e^{-2T}$.

It is shown in Appendix 3 that $\underline{H}^*(z)$ of the third order PWM System belonging to the class $\Gamma_n(\underline{K}, 0, \underline{K}')$ is given by

$$\underline{H}^*(z) = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix} > 0, \quad |z|=1, \qquad (4.3.9)$$

where the elements of $\underline{H}^*(z)$ are given by

$$H_{11} = \frac{2}{KAb} - 0.5 + \frac{2Q_1(1-\cos\theta)}{KAb} + Q_1 \cos\theta,$$

$$H_{22} = \frac{2T}{KAb(1-p)} + \frac{2Q_2(1-\cos\theta)}{KAb} - \frac{1}{1+p^2-2p\cos\theta}\Big[2Q_2\sin^2\theta + 2(\cos\theta - p)\{1+Q_2(1-\cos\theta)\}\Big],$$

$$H_{33} = \frac{4T}{KAb(1-q)} + \frac{2Q_3(1-\cos\theta)}{KAb} + \frac{1}{1+q^2-2q\cos\theta}\Big[Q_3\sin^2\theta + (\cos\theta - q)\{1+Q_3(1-\cos\theta)\}\Big],$$

$$H_{12} = 0.5Q_2(\text{Cos }\theta + j\text{Sin }\theta) + \frac{0.5(\text{Cos }\theta - 1 + j\text{ Sin }\theta)}{2(1-\text{Cos }\theta)} -$$

$$\frac{\text{Cos }\theta - p - j\text{ Sin }\theta}{1+p^2-2p\text{ Cos }\theta}\left\{1+Q_1(1-\text{Cos }\theta + j\text{ Sin }\theta)\right\},$$

$$H_{13} = 0.5Q_3(\text{Cos }\theta + j\text{ Sin }\theta) + \frac{0.5(\text{Cos }\theta - 1 + j\text{ Sin }\theta)}{2(1-\text{Cos }\theta)} +$$

$$\frac{0.5(\text{Cos }\theta - q - j\text{ Sin }\theta)}{1+q^2-2q\text{ Cos }\theta}\left\{1+Q_1(1-\text{Cos }\theta + j\text{ Sin }\theta)\right\},$$

$$H_{23} = \frac{0.5(\text{Cos }\theta - q - j\text{ Sin }\theta)}{1+q^2-2q\text{ Cos }\theta}\left\{1+Q_2(1-\text{Cos }\theta + j\text{ Sin }\theta)\right\} -$$

$$\frac{(\text{Cos }\theta - p + j\text{ Sin }\theta)}{1+p^2-2p\text{ Cos }\theta}\left\{1+Q_3(1-\text{Cos }\theta - j\text{ Sin }\theta)\right\},$$

$$H_{21} = \overline{H_{12}},$$

$$H_{31} = \overline{H_{13}},$$

$$H_{32} = \overline{H_{23}}.$$

From the enequality (4.3.9) and the properties of a positive definite matrix[18], we must have,

$$\begin{aligned} H_{11} &> 0, \\ H_{22} &> 0, \\ H_{33} &> 0, \\ H_{11}\ H_{22} - H_{21}\ H_{12} &> 0, \\ \left|\underline{H}^*(s)\right| &> 0. \end{aligned} \qquad (4.3.10)$$

Evaluation of $\underline{H}^*(s)$ results in a cubic equation in (KAb) which must be maximized with respect to Q_j (j=1,2,3). Thus the critical loop gain is given by

$$(KAb)_c = \max_{Q_j} (KAb), \text{ such that } |H^*(z)| \geq 0, \quad j = 1, 2, 3$$

A computer program was written on IBM7044 to obtain the critical loop gain of the second and third order systems. The critical loop gain obtained for the third order system for different frequencies is plotted in Fig.7.

FREQUENCY OF OSCILLATION

TABLE 1. SECOND ORDER SYSTEM

Sampling frequency in cps.	Frequency of oscillation in rps.	
	Experiment	Describing function method
0.5	1.57	1.6
1.0	2.08	1.7
2.0	2.10	2.2
3.0	2.52	2.6
4.0	2.62	2.9

TABLE 2. THIRD ORDER SYSTEM

Sampling frequency in cps.	Frequency of oscillation in rps.	
	Experiment	Describing function method
0.5	1.57	0.8
1.0	1.00	0.9
2.0	1.10	1.2
3.0	1.05	1.3
4.0	1.05	1.4

CRITICAL LOOP GAIN $(KAb)_c$

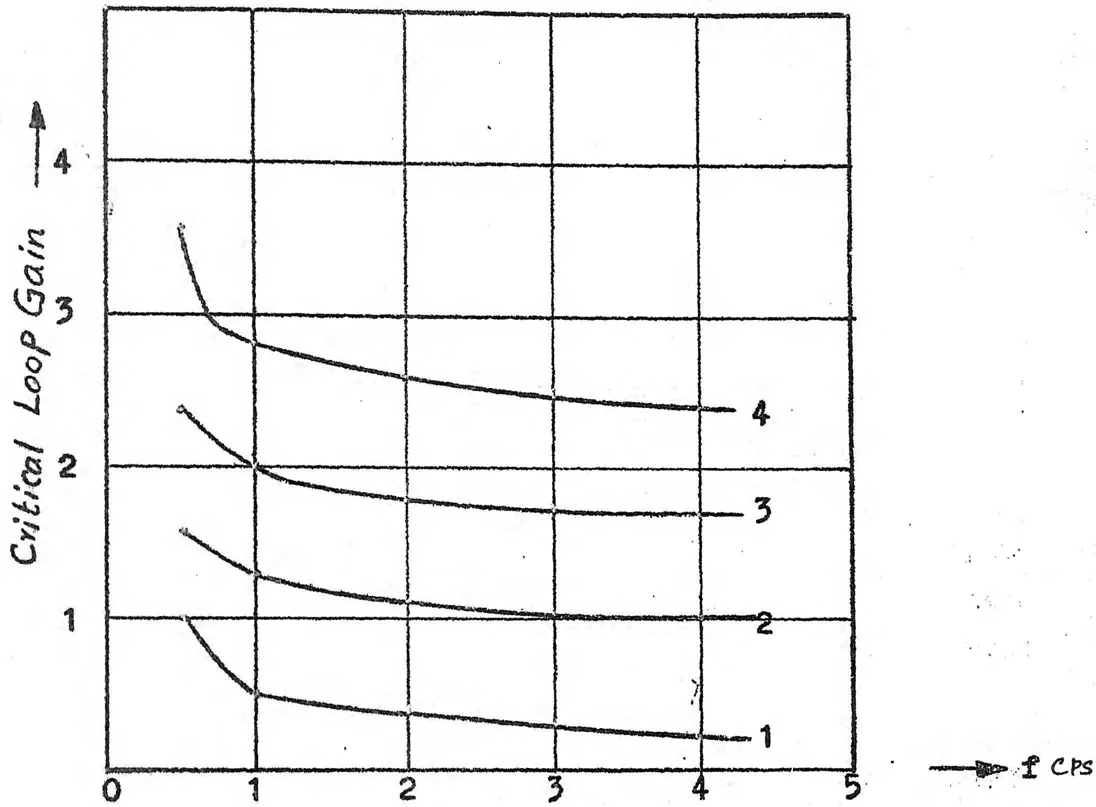
TABLE 3. SECOND ORDER PWM SYSTEM

Frequency cps.	Experiment	Lyapunov's Method	Describing Function Method	Popov's Method
0.5	2.45	1.0	3.58	1.52
1.0	1.99	0.5	2.85	1.25
2.0	1.80	0.4	2.65	1.16
3.0	1.75	0.26	2.50	1.10
4.0	1.70	0.21	2.47	1.00

CRITICAL LOOP GAIN $(KAb)_c$

TABLE 4. THIRD ORDER PWM SYSTEM

Frequency cps.	Experiment	Lyapunov's Method	Describing Function Method	Popov's Method
0.5	5.00	1.30	6.05	2.47
1.0	2.00	0.76	2.85	1.34
2.0	1.40	0.40	1.90	1.25
3.0	1.15	0.30	1.50	1.00
4.0	0.80	0.25	1.37	0.67



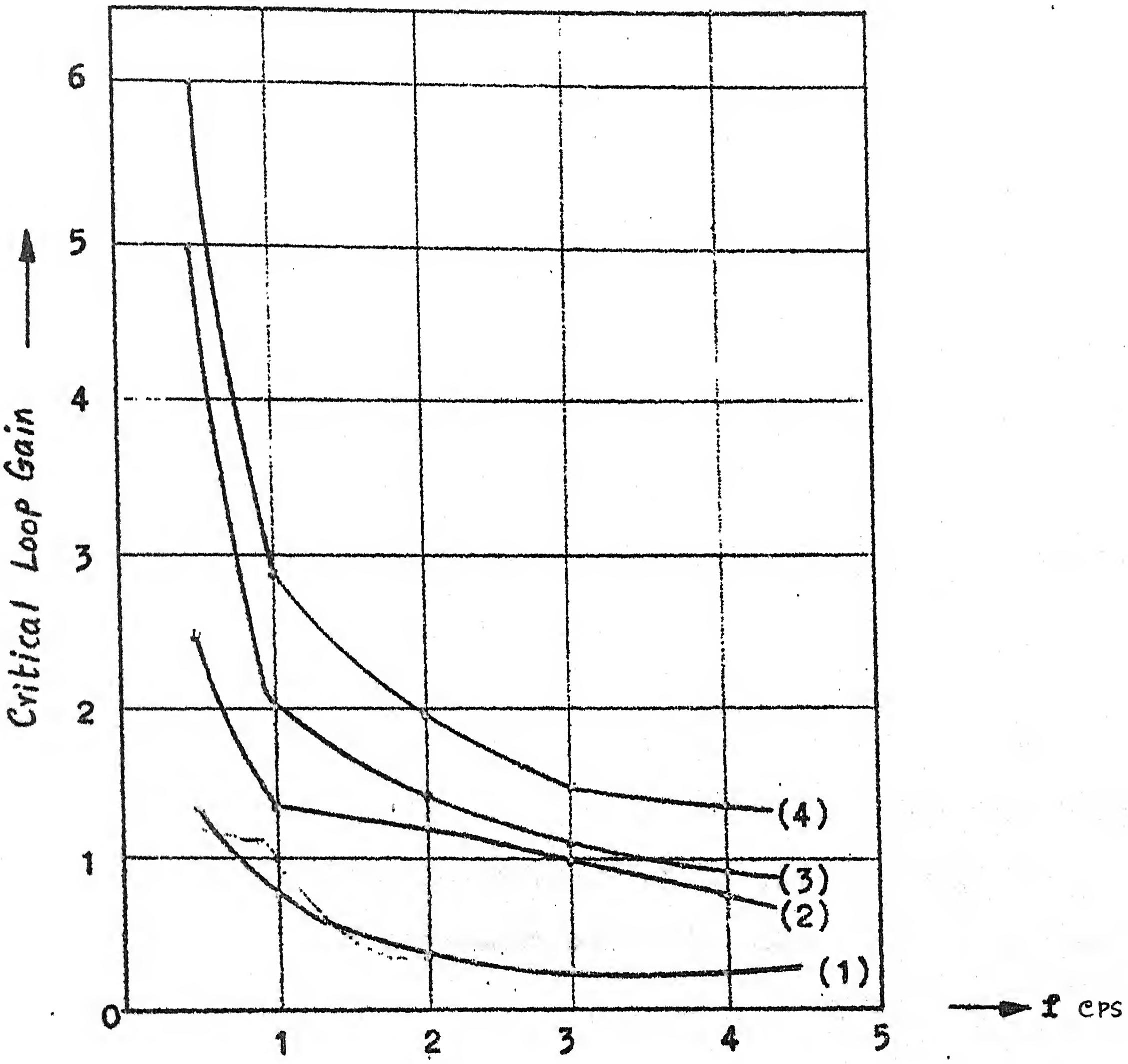
(1) Lyapunov's Method

(2) Popov's Method - $\Gamma_n(\underline{K}, 0, \underline{K}')$

(3) Experimental Simulation

(4) Describing Function Method.

FIG. 6: CRITICAL LOOP GAIN Vs. FREQUENCY CURVES OF SECOND ORDER PWM SYSTEM.



(1) Lyapunov's Method

(2) Popov's Method $\Gamma_n(\underline{K}, 0, \underline{K}')$

(3) Experimental Simulation

(4) Describing Function Method.

FIG. 7: CRITICAL LOOP GAIN Vs. FREQUENCY CURVES OF THIRD ORDER PWM SYSTEM.

CHAPTER - V

SUMMARY AND CONCLUSIONS

In the present work, the stability boundaries for second and third order systems are obtained through 1) Lyapunov's method, 2) Experimental simulation, 3) Describing function technique and 4) Popov's method. The critical loop gains obtained are plotted against sampling frequencies in Figs. 6 and 7.

Lyapunov's method deals with the stability of the trivial solution of a system of difference equations through the use of the Lyapunov's function and the stability that is of interest to us is asymptotic stability in the large of an autonomous system of difference equations. The conditions for asymptotic stability obtained through Lyapunov's function as well as Popov's method are sufficient, but not necessary and sufficient. In the case of Lyapunov's method, the degree of necessity of the obtained condition depends on the appropriateness of the choice of a Lyapunov's function for a given system and one way of improving the necessity of the condition is to choose a more appropriate Lyapunov's function for the system. However, for the PWM System, the choice of a Lyapunov's function is limited in order to get an analytically feasible condition for stability. Another possibility of improvement may be to obtain a condition for instability through the use of the theorems on instability in the second method of Lyapunov. In addition to finding the stability boundaries, the method also provides a way of finding the parameters of the system.

Popov's method is essentially a frequency domain approach and although this method yields less satisfactory stability regions, there is a considerable amount of improvement over the boundaries obtained through Lyapunov's method. Furthermore, when an upper bound for the a.c. gain of the non-linearities is given, it is found that the stability boundaries obtained are close to the experimental values. Thus Popov's method yields less conservative results as compared to the Lyapunov's method. The great advantage of the frequency domain inequality is obviously its ease of application. The method also needs no knowledge of the type of non-linearities present in the system except the bounds on d.c. and a.c. gains.

The describing function technique is partly a graphical method and the curves are plotted at various frequencies for all magnitudes of inputs to the non-linearities in order to find the critical loop gain and the existence of limit cycles. Thus the method is quite involved as compared to Popov's method and is less cumbersome to that of Lyapunov's method. However, the method is useful in finding the stability boundaries as well as the existence of limit cycles about which Lyapunov's method and Popov's method fail to provide any information. The results obtained in the describing function technique are pessimistic and this may be due to the assumption of adequate filtering of harmonics and the approximations involved in the analysis.

In conclusion, it is to be pointed out that Popov's method gives results which are very close to the experimental values and is less conservative to that of Lyapunov's method. Furthermore, the frequency of oscillations obtained in describing function technique compares favourably with the experimentally found frequency of oscillations. A further improvement in the results of Popov's method is possible if Steepest-descent method is used to find the critical loop gain with all the constraints taken into account.

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      GENERAL PROGRAMME FOR DESCRIBING FUNCTION ANALYSIS OF PWM.
      FORTRAN FOUR ONLY
      N..ORDER OF THE SYSTEM, B..PWM CONSTANT
      A(I)..POLES OF THE SYSTEM
      T(I)..SAMPLING PERIODS
      P(I)..PARTIAL FRACTION COEFFICIENTS
      DIMENSION T(8),A(8),PN(8),P(8)
      COMPLEX AA,A1,G1,G
62    READ 1,N,B
1     FORMAT(I2,F10.0)
      READ2,(A(I),I=1,N)
      READ2,(T(I),I=1,8)
      READ2,(P(I),I=1,N)
2     FORMAT(8F1 .0)
      PM=4.*ATAN(1.)
      DO 10 I=1,N
      E=0.0
      EP=T(I)/B
35    IF(E-EP)7,7,3
7     DO 20 J=1,N
      Y1=B*E*A(J)
      Y2=Y1*Y1
      Y3=Y2*Y1
      Y4=Y3*Y1
      X=EXP(A(J)*T(I))
      PN(J)=X*(1.-0.425*Y1+0.125*Y2-0.0265*Y3+0.0104*Y4)
20    CONTINUE
      GO TO 5
3     S1=ARSIN(T(I)/(B*E))
      S2=SIN(2.0*S1)/2.0
      Q1=2. *(S1 S2)/PM
      DO 30 J=1,N
      X=EXP(A(J)*T(I))
      IF(X-1.)31,33,31
33    S3=0.
      GO TO 32
31    S3=(1.0+(1.0-X)/(A(J)*T(I)))/E
32    S4=ARSIN(S3)
55    S5=SIN(2.0*S4)/2.0
      Q2=(1.0-X)*(S4+S5)*2.0/PM
      PN(J)=Q1-Q2
30    CONTINUE
5     W=.2
6     C=W*T(I)
      D=0.0
      AA=CMPLX(D,C)
      A1=CEXP(AA)
      G=0.0
      DO 50 J=1,N
      X=EXP(A(J)*T(I))
      G1=PN(J)*P(J)/(A1-X)
      G=G+G1
50    CONTINUE
```

```
      PRINT100,T(I),W,E,G
  100 FORMAT(3F12.3,(F10.4,F10.4))
      W=W+0.2
      IF(2. -W)4,6,6
    4 W=0.0
      IF(E-2.0*EP)25,25,10
   25 E=E+0.25*EP
      GO TO 35
   10 CONTINUE
      GO TO 62
   70 STOP
      END
```

APPENDIX - II

Popov's Method for a Second Order System

For the system given by

$$G(s) = \frac{K}{s(s+1)} \tag{2.1}$$

$$\underline{G}^*(z) = \begin{bmatrix} \frac{K}{z-1} & \frac{-K}{z-p} \\ \frac{K}{z-1} & \frac{-K}{z-p} \end{bmatrix} \tag{2.2}$$

The diagonal matrices $\underline{Q}$, $\underline{K}$ and $\underline{K}'$ are given by

$$\underline{Q} = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} ; \quad \underline{K} = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \quad \text{and}$$

$$\underline{K}' = \begin{bmatrix} K_1' & 0 \\ 0 & K_2' \end{bmatrix} \tag{2.3}$$

From equation (4.2.6) of theorem 3,

$$\underline{X}_3^*(z) = \begin{bmatrix} \frac{K}{z-1}\left(1+ \frac{z-1}{z}Q_1\right)+\frac{1}{2}\left|z-1\right|^2 \frac{Q_1}{K_1'} & \frac{-K}{z-p}\left(1+ \frac{z-1}{z} Q_1\right) \\ \frac{K}{z-1}\left(1 + \frac{z-1}{z} Q_2\right) & \frac{1}{2}\left|z-1\right|^2 \frac{Q_2}{K_2'} - \frac{K}{z-p}\left(1+ \frac{z-1}{z} Q_2\right) \end{bmatrix} \quad (2$$

Using equation (2.4) and its complex conjugate transpose, we obtain

$$\underline{H}^*(z) = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \tag{2.5}$$

where the elements of $\underline{H}^*(z)$ are given as follows:

$$H_{11} = \frac{2}{K_1} + |z-1|^2 \, \frac{Q_1}{K_1'} + \frac{K}{z-1} + \frac{KQ_1}{z} + \frac{Kz}{1-z} + KzQ_1 \;,$$

$$H_{12} = \frac{-K}{z-p} - \frac{K(z-1)Q_1}{z(z-p)} + \frac{Kz}{1-z} + KzQ_2 \;,$$

$$H_{21} = \frac{K}{z-1} + \frac{KQ_2}{z} - \frac{Kz}{1-pz} - \frac{z(1-z)}{1-pz}\, Q_1 K \;,$$

$$H_{22} = \frac{2}{K_2} + |z-1|^2 \, \frac{Q_2}{K_2'} - \frac{K}{z-p} - \frac{(z-1)Q_2K}{z(z-p)} - \frac{Kz}{1-pz} - \frac{K(1-z)z}{1-pz}\, Q_2 \,.$$

From equations (4.2.9) and (4.2.10),

$$K_1 = \frac{V_1(e)}{e} = Ab \;,$$

$$K_2 = \frac{V_2(e)}{e} = \underline{\frac{Ab(1-p)}{T}} \;, \tag{2.6}$$

$$K_1' = K_2' = Ab.$$

Substitution of equations (2.6) into (2.5) and putting $z = \cos\theta + j\sin\theta$, we obtain the matrix inequality given in (4.3.4).

APPENDIX - III

Absolute Stability of Third Order System

The transfer matrix of a third order linear time invariant plant is given by

$$\underline{G}^*(z) = \begin{bmatrix} \frac{0.5K}{z-1} & \frac{-K}{z-p} & \frac{0.5K}{z-q} \\ \frac{0.5K}{z-1} & \frac{-K}{z-p} & \frac{0.5K}{z-q} \\ \frac{0.5K}{z-1} & \frac{-K}{z-p} & \frac{0.5K}{z-q} \end{bmatrix} \qquad (3.1)$$

with $p = e^{-T}$, $q = e^{-2T}$.

Defining $\underline{Q}$, $\underline{K}$ and $\underline{K}'$ matrices as detailed in Appendix II, we get the following:

$$\underline{I} + \left(\frac{z-1}{z}\right)\underline{Q}\ \ \underline{G}^*(z)$$

$$= \begin{bmatrix} \frac{0.5K}{z-1}(1+Q_1\frac{z-1}{z}) & \frac{-K}{z-p}(1+Q_1\ \frac{z-1}{z}) & \frac{0.5K}{z-q}(1+Q_1\ \frac{z-1}{z}) \\ \frac{0.5K}{z-1}(1+Q_2\frac{z-1}{z}) & \frac{-K}{z-p}(1+Q_2\ \frac{z-1}{z}) & \frac{0.5K}{z-q}(1+Q_2\ \frac{z-1}{z}) \\ \frac{0.5K}{z-1}(1+Q_3\frac{z-1}{z}) & \frac{-K}{z-p}(1+Q_3\ \frac{z-1}{z}) & \frac{0.5K}{z-q}(1+Q_3\ \frac{z-1}{z}) \end{bmatrix} \qquad (3.2)$$

$$\frac{1}{2}|z-1|^2\ \ \underline{Q}\ \underline{K}'^{-1} = \frac{\frac{1}{2}|z-1|^2}{Ab} \begin{bmatrix} Q_1 & 0 & 0 \\ 0 & Q_2 & 0 \\ 0 & 0 & Q_3 \end{bmatrix} \qquad (3.3)$$

Addition of (3.2) and (3.3) gives $\underline{Y}_3^*(z)$.

Then $\underline{H}^*(z)$ can be expressed as

$$\underline{H}^*(z) = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix} \qquad (3.4)$$

where the elements of $\underline{H}^*(z)$ are given by

$$H_{11} = \frac{2}{KK_1} + \frac{0.5}{z-1} + \frac{0.5Q_1}{z} + \frac{0.5z}{1-z} + 0.5\ zQ_1 + \frac{|z-1|^2\ Q_1}{KAb},$$

$$H_{22} = \frac{2}{KK_2} - \frac{1}{z-p} - \frac{(z-1)Q_2}{z(z-p)} - \frac{z}{1-pz} - \frac{z(1-z)Q_2}{1-pz} + \frac{|z-1|^2\ Q_2}{KAb},$$

$$H_{33} = \frac{2}{KK_3} + \frac{0.5}{z-q} + \frac{0.5(z-1)Q_3}{z(z-q)} + \frac{0.5z}{1-qz} + \frac{0.5z(1-z)Q_3}{1-qz} + \frac{|z-1|^2 Q_3}{KAb},$$

$$H_{12} = 0.5zQ_2 + \frac{0.5z}{1-z} - \frac{1}{z-p} - \frac{(z-1)Q_1}{z(z-p)},$$

$$H_{13} = \frac{0.5}{z-q} + \frac{0.5(z-1)Q_1}{z(z-q)} + \frac{0.5z}{1-z} + 0.5zQ_3,$$

$$H_{23} = \frac{0.5}{z-q} + \frac{0.5(z-1)Q_2}{z(z-q)} - \frac{z}{1-pz} - \frac{z(1-z)Q_3}{1-pz}.$$

Substituting $z = \cos\theta + j\sin\theta$, H_{21}, H_{31}, and H_{32} are found to be complex conjugate of H_{12}, H_{13} and H_{23} respectively.

From equations (4.2.9) and (4.2.10),

$$K_1 = \frac{V_1(e)}{e} = Ab,$$

$$K_2 = \frac{V_2(e)}{e} = \frac{Ab(1-p)}{T} \quad \text{and} \qquad (3.5)$$

$$K_3 = \frac{V_3(e)}{e} = \frac{Ab(1-q)}{2T}.$$

Substitution of (3.5) into (3.4) results in equation (4.3.9) with all the elements as specified.

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